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$\mathcal{H}$ −LOCALLY CLOSED SETS IN GENERALIZED TOPOLOGICAL SPACE

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Abstract

The aim of this paper is extend the study of $\mathcal{H}$ −locally closed sets are defined in a generalized topological space with a hereditary class, characterize this sets and discussits properties and we discuss the properties of $\mathcal{H}_A$ −sets, $\mathcal{H}_B$ −sets, $\mathcal{H}_C$ −sets and $f_{\mathcal{H}}$ −sets.It is established that $\mathcal{H}_A$ −sets are $\mathcal{H}$ −locally closed set and $\mathcal{H}_B$ − sets are $\mathcal{H}_C$ −sets in generalized topological space.

Keywords: Generalized topological space, hereditary class, κ^*−dense set, $semi^* - \mathcal{H}$ −open set.

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1. Introduction

Let X be a nonempty set. A non-empty subfamily κ of $\wp(X)$ is called a generalized topologyon X [1] if $\emptyset \in \kappa$ and κ is closed under arbitrary union. The pair (X,κ) is called generalized topological space. Elements of κ are called κ −open sets and the complement of a κ −open setis called a κ −closed set. The largest κ −open set contained in a subset A of X is denoted by $int_\kappa(A)$[3] and is called the κ −interior of A. The smallest κ −closed set containing A is called the κ − closure of A and is denoted by $cl_\kappa(A)$ [3].

Throughout the paper, by a space, we always mean a generalized topological space (X,κ).A subset A is said to be κ −dense if $cl_\kappa(A) = X$. A hereditary class $\mathcal{H}$ is a non empty family ofsubset of X such that $A \subset B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$ [2]. For each subset A of X, a subset $A^*(\mathcal{H})$ or simply A^* of X is defined by $A^*(\mathcal{H}) = \{x \in X \mid M \cap A \notin \mathcal{H}$ for every $M \in \kappa$ containing $X\}$[2]. A generalized topology κ is said to be a quasi-topology [4] on X if $M, N \in \kappa$ implies $M \cap N \in \kappa$.If $cl_\kappa^*(A) = A \cap A^*$ for every subset A of X, with respect to κ and a hereditary class $\mathcal{H}$ of subsets of X then $\kappa^* = \{A \subset X / cl_\kappa^*(X - A) = X - A\}$ is a generalized topology[2]. Elements of κ^* are called κ^* −open sets and the complement of a κ^* −open set is called a κ^* −closed set. $int_\kappa^*(A)$ is the interior of A in (X,κ^*). Let (X,κ) be a generalized topological space and $\mathcal{H}$ be a hereditary class of subsets of X. If $cl_\kappa^*(A) = X$, then A is called κ^*-dense.

Definition 1.1. *Let (X,κ) be a generalized topology. A subset A of X is said to be*

(i) κ − *regular closed [2] if* $cl_\kappa(int_\kappa(A)) = A$,

(ii) κ −*semi open[2] if* $A \subset cl_\kappa(int_\kappa(A))$,

(iii) $\kappa - \alpha -$ *open [2] if* $A \subset int_\kappa(cl_\kappa(int_\kappa(A)))$,

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(iv) $\kappa-\beta-$ *open[2] if* $A \subset cl_\kappa(int_\kappa(cl_\kappa(A)))$,

(v) $\beta-\mathcal{H}-$ *open[3] if* $A \subset cl_\kappa(int_\kappa(cl^*_\kappa(A)))$.

(vi) $\alpha-\mathcal{H}-$ *open[3] if* $A \subset int_\kappa(cl^*_\kappa(int_\kappa(A)))$,

(vii) $pre-\mathcal{H}-$ *open [3] if* $A \subset int_\kappa(cl_\kappa(A))$,

(viii) $semi-\mathcal{H}-$ *open[3] if* $A \subset cl^*_\kappa(int_\kappa(A))$,

(ix) $\kappa-preopen$ [3] if $A \subset int_\kappa(cl_\kappa(A))$.

The family of all $\kappa-semi\ open$ ($\kappa-\beta-open$, $\alpha-\mathcal{H}-open$) sets is denoted by $\sigma_\kappa(\beta_\kappa, \alpha\mathcal{H}o(X))$. If $\mu \in \{\kappa, \sigma, \beta, \pi\}$, then int_μ and cl_μ are respectively, the interior and closure operators with respect to the generalized topology μ.

The complement of a $\kappa-semi\ open$ ($resp.$ $\kappa-pre\ open$, $\kappa-\alpha-open$, $\kappa-\beta-open$, $\beta-\mathcal{H}-open$, $\alpha-\mathcal{H}-open$, $pre-\mathcal{H}-open$, $semi-\mathcal{H}-open$) set is said to be a $\kappa-semi\ closed$ ($resp.$ $\kappa-pre\ closed$, $\kappa-\alpha-closed$, $\kappa-\beta-closed$, $\beta-\mathcal{H}-closed$, $\alpha-\mathcal{H}-closed$, $pre-\mathcal{H}-closed$, $semi-\mathcal{H}-closed$) sets.

Lemma 1.2. *[8] Let* (X,κ) *be a generalized topological space and* $\mathcal{H}$ *be a hereditary class of subsets of X. Then* $\mathcal{H}$ *is* $\kappa-codense$ *if and only if* $A \subset A^*$ *for every* $A \in \kappa$.

Lemma 1.3. *[3] Let* (X,κ) *be a generalized topological space and* $\mathcal{H}$ *be a hereditary class of subsets of* X. *If* $A \subset A^*$, *then* $A^* = cl_\kappa(A) = cl^*_\kappa(A)$.

Lemma 1.4. *[8] Let* (X,κ) *be a generalized topological space with hereditary class* $\mathcal{H}$. *Then the following are equivalent.*

(a) $\mathcal{H}$ *is strongly* $\kappa-codense$.

(b) $M \subset M^*$ *for every* $M \in \kappa$.

(c) $S \subset S^*$ *for every* $S \in \sigma(X)$.

(d) $cl_\kappa(M) = M^*$ *for every* $M \in \kappa$.

Lemma 1.5. *[3] Let* (X,κ) *be a generalized topological space and* $A \subset X$. *Then the following hold.*

(a) $int_\pi(A) = A \cap int_\kappa(cl_\kappa(A))$.

(b) $cl_\pi(A) = A \cup cl_\kappa(int_\kappa(A))$.

Lemma 1.6. *Let* (X,κ) *be a quasi-topological space and* $\mathcal{H}$ *be a hereditary class of subsets of* X. *If* $A \subset X$ *and* $M \in \kappa$, *then the following hold.*

(a) $M \cap A^* \subset (M \cap A)^*$ *[8, Theorem 2.6].*
(b) $M \cap cl_\kappa(A) \subset cl_\kappa(M \cap A)$ *[8, Lemma 1.3].*

Lemma 1.7. *[3] Let* (X,κ) *be a generalized topological space,* $\mathcal{H}$ *be a hereditary class of subsets of* X *and* $A \subset B \subset X$. *Then the following hold.*

(a) $A^* \subset B^*$.

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(b) $A^* \subset cl_\kappa(A)$.

(c) $G \in \kappa$, $G \cap A \in H$ *implies that* $G \cap A^* = \emptyset$. *Hence* $\mathcal{H}^* = X - G$ *if* $\mathcal{H}^* \in \mathcal{H}$.

(d) A^* *is* $\kappa -$*closed for* $A \subset X$.

(e) *If* F *is* $\kappa -$*closed, then* $F^* \subset F$.

(f) $A^{**} = (A^*)^* \subset A^*$ *for every* $A \subset X$.

(g) $X = X^*$ *if and only if* $\kappa \cap \mathcal{H} = \{\emptyset\}$.

2. $\mathcal{H} -$ Locally closed sets

Let (X, κ) *be a generalized topological space,* $\mathcal{H}$ *be a hereditary class of subsets of* X *and* A *be a subset of* X. *Then* A *is* $\mathcal{H}$ *–locally closed [2] if* $A = G \cap V$, *where* G *and* V *are* κ^* *–perfect. The following Theorem 2.1 gives a characterization of* $\mathcal{H}$ *–locally closed sets.*

Theorem 2.1. *Let* (X, κ) *be a quasi-topological space and* $\mathcal{H}$ *be a hereditary class of subsets of* X. *Then the following are equivalent.*

(a) A *is a* $\mathcal{H}$ *–locally closed set.*

(b) $A = U \cap A^*$ *for some* $\kappa - open\ set\ U$.

Proof: $(a) \Rightarrow (b)$. If A is a $\mathcal{H}$ –locally closed set, then $A = U \cap V$ where U is $\kappa - open$ and V is κ^* –perfect. $A = U \cap V$ implies that $A^* = (U \cap V)^*$. By Lemma 1.6(a), $(U \cap V)^* \supset U \cap V^*$. V is κ^* –perfect, implies that $A^* = (U \cap V)^* \supset U \cap V^* = U \cap V = A$ which implies that $A \subset A^*$. Thus $A \subset V$ implies that $A^* \subset V^* = V$ and so $A^* = A^* \cap V$. Hence $U \cap A^* = U \cap (A^* \cap V) = U \cap V \cap A^* = A \cap A^* = A$.

(b)⇒(a). Conversely, suppose that $A = U \cap A^*$ for some κ –open set U. Since $A \subset A^*$, byLemma 1.7, $A^* \subset A^{**} \subset A$. Therefore A^* is κ^* –perfect and so A is an $\mathcal{H}$ –locally closed set.

The following Example 2.2 shows that the condition quasi topology on κ *cannot be droppedin Theorem 2.1.*

Example 2.2. Consider the space (X, κ) with the hereditary class $\mathcal{H}$ where $X = \{1,2,3,4\}$, $\kappa = \{\emptyset, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ and $H = \{\emptyset, \{2\}, \{4\}\}$. Clearly, κ is not a quasi-topology. $If\ A = \{2,3\}$, then $A^* = \{3,4\}$. Clearly, A is $\mathcal{H} -$ locally closed set. But for every κ –open set U, $A \neq U \cap A^*$.

The following Theorem 2.3 gives a characterizations of $\mathcal{H}$ *–locally closed sets.*

Theorem 2.3. *Let* (X, κ) *be a quasi-topological space,* $\mathcal{H}$ *be a hereditary class of subsets of* X *and* A *be a subset of. Then the following are equivalent.*

(a) A *is* $\mathcal{H}$ *–locally closed.*

(b) $A = G \cap A^*$ *for some* κ *–open set* G.

(c) $A \subset A^*$ *and* $A^* - A$ *is* κ *–closed.*

(d) $A \subset A^*$ *and* $A \cap (X - A^*)$ *is* κ *–open.*

(e) $A \subset A^*$ *and* $A \subset int(A \cap (X - A^*))$.

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Proof: $(a) \Rightarrow (b)$. Follows from Theorem 2.1.

$(b) \Rightarrow (c)$. Suppose $A = G \cap A^*$ for some κ −open set G. Clearly, $A \subset A^*$ and $A^* - A = A^* \cap (X - A) = A^* \cap (X - (G \cap A^*)) = A^* \cap (X - G)$. Since A^* is κ −closed, by Lemma 1.7(d), $A^* - A$ is κ −closed.

$(c) \Rightarrow (d)$. $A^* - A$ is κ −closed implies that $A^* \cap (X - A)$ is κ −closed which implies that $X - (A^* \cap (X - A))$ is κ−open . Therefore $A \cup (X - A^*)$ is κ −open.

$(d) \Rightarrow (e)$ is clear.

$(e) \Rightarrow (a)$. $X - A^* = int_\kappa(X - A^*) \subset int_\kappa(A \cup (X - A^*))$ and so $A \cup (X - A^*)$ is κ −open, by hypothesis. Since $A = (A \cup (X - A^*)) \cap A^*$, A is $\mathcal{H}$ −locally closed.

Corollary 2.4. *Let (X, κ) be a quasi-topological space, $\mathcal{H}$ be a hereditary class of subsets of X and A be a subset of. Then the following are equivalent.*

i. A is κ^* −dense in itself.

ii. A is κ^* −perfect.

iii. $cl_\kappa(A) = cl^*_\kappa(A) = A^*$.

Proof: *i.* Follows from Theorem 2.3.

ii. Follows from (a) and the fact that $(A^*)^* \subset A^*$.

iii. Follows from Lemma 1.3.

The following Theorem 2.5 gives a characterization of $\mathcal{H}$ −locally closed sets.

Theorem 2.5. *Let (X, κ) be a quasi-topological space, $\mathcal{H}$ be a hereditary class of subsets of X and A be a subset of X. Then A is $\mathcal{H}$ −locally closed if and only if A is locally κ −closed andA is κ^* −dense in itself.*

Proof: If A is $\mathcal{H}$ −locally closed, then A is κ^* −dense in itself and $A = G \cap A^*$ for some $G \in \kappa$. Since A^* is κ −closed, A is locally κ −closed. Conversely, if A is locally κ −closed, then $A = G \cap F$ where $G \in \kappa$ and F is κ −closed. $A \subset F \Rightarrow A^* \subset F \Rightarrow A^* \cap F = A^*$. Now A is κ^* −dense in itself implies that $A \subset A^*$ and so $A = A \cap A^* = (G \cap F) \cap A^* = G \cap (F \cap A^*) = G \cap A^*$. Therefore, A is $\mathcal{H}$ − locally closed.

The following Example 2.6 shows that κ^* −dense in itself sets need not be $\mathcal{H}$ −locally closed.

Example 2.6. *Let* $X = \{e, f, g, h\}$, $\kappa = \{\emptyset, \{h\}, \{e, g\}, \{e, g, h\}, X\}$ *and* $\mathcal{H} = \{\emptyset, \{g\}, \{h\}, \{g, h\}\}$. If $A = \{e\}$, then A is κ^* −dense itself but not $\mathcal{H}$ −locally closed.

Definition 2.7. *Let (X, κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and A be a subset of X is said to be an $\alpha_{\mathcal{H}N}$ − set if $A = U \cap V$ where U is $\alpha - \mathcal{H}$ −open,and V is κ^* −closed. We will denote the family of all $\alpha_{\mathcal{H}N}$ −sets denoted by $\alpha_{\mathcal{H}N}(A)$.*

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Definition 2.8. *Let (X,κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and A be a subset of X. A is said to be a weakly $\mathcal{H}-$ locally closed set if $A = U \cap V$ where U is $\kappa-$open and V is a κ^*-closed set. The family of all weakly $\mathcal{H}-$ locally closed setsis denoted by $W\mathcal{H}LC(X)$.*

Clearly, every weakly $\mathcal{H}-$ locally closed set is an $\alpha_{\mathcal{H}N}-$set but not the converse as shown bythe following Example 2.9.

Example 2.9. *Consider (X,κ) the quasi topological space where $X = \{p, q, r\}$, $\kappa = \{\emptyset,\{p\},\{p,r\},X\}$ and $H = \{\emptyset,\{q\},\{r\},\{q,r\}\}$. If $A = \{p,q\}$, then $int_\kappa\left(cl_\kappa^*(int_\kappa(A))\right) = int_\kappa\left(cl_\kappa^*(int_\kappa(p,q))\right) = int_\kappa(cl_\kappa^*(p)) = int_\kappa(p,q,r)$ $= X \supset A$ and so A is $\alpha-\mathcal{H}-$open and hence an $\alpha_{\mathcal{H}N}-$set. But there is no $\kappa-$open set U such that $A = U \cap cl_\kappa^*(A)$ where $cl_\kappa^*(A) = X$. Hence A is not a weakly $\mathcal{H}-$locally closed set.*

Theorem 2.10 below gives a characterization of $\alpha_{\mathcal{H}N}-$sets.

Theorem 2.10. *Let (X,κ) be a quasi-topological space, $\mathcal{H}$ be a hereditary class of subsets of X and A be a subset of X. Then A is $\alpha_{\mathcal{H}N}-$set if and only if $A = U \cap cl_\kappa^*(A)$ for some $\alpha_{\mathcal{H}N}-$set U.*

Proof: If A is an $\alpha_{\mathcal{H}N}-$set, then $A = U \cap V$ where U is $\alpha-\mathcal{H}-$open and V is κ^*-closed. Since $A \subset V$, $cl_\kappa^*(A) \subset cl_\kappa^*(V) = V$ and so $cl_\kappa^*(A) \subset U \cap V = A \subset U \cap cl_\kappa^*(A)$ which implies that $A = U \cap cl_\kappa^*(A)$.

Conversely, suppose $A = U \cap cl_\kappa^*(A)$ for some $\alpha_{\mathcal{H}N}-$ set U. Since $cl_\kappa^*(A)$ is κ^*-closed, A is an $\alpha_{\mathcal{H}N}-$set.

In the following Theorem 2.11, we give the relation of κ^*-perfect, $\kappa-$locally closed and κ^*-dense in itself subsets with $\mathcal{H}-$locally closed subsets.

Theorem 2.11. *Let (X,κ) be a quasi-topological space, $\mathcal{H}$ be a hereditary class of subsets of X and A be a subset of X. If A is κ^*-perfect, then A is $\mathcal{H}-$locally closed. The converse is true, if A is κ^*-closed.*

Proof: If A is κ^*-perfect, then $A = A^*$ and so $A = X \cap A = X \cap A^*$ which implies that A is $\mathcal{H}-$locally closed.

Conversely, if A is $\mathcal{H}-$locally closed, then $A \subset A^*$. A is κ^*-closed implies that $A^* \subset A$. Hence $A = A^*$.

3. $\mathcal{H}_{A1}$ and $\mathcal{H}_{B1}$ sets

Definition 3.1. *Let (X,κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and A be a subset of X, A is said to be a $\mathcal{H}_{A1}-$set if $A = U \cap V$ where U is $\kappa-$openand $cl_\kappa^*(int_\kappa(V)) = X$. The family of all $\mathcal{H}_{A1}-$sets is denoted by $\mathcal{H}_{A1}(A)$.*

Definition 3.2. *Let (X,κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and A be a subset of X. A is said to be a $\mathcal{H}_{B1}-$set if $A = U \cap V$ where U is $\alpha-\mathcal{H}-$open and $cl_\kappa^*(int_\kappa(V)) = X$. The family of all $\mathcal{H}_{B1}-$sets is denoted by $\mathcal{H}_{B1}(X)$. Clearly, $\mathcal{H}_{A1}(X) \subset \mathcal{H}_{B1}(X)$.*

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The following Theorem 3.3 shows that $\mathcal{H}_{A1}$ –sets and $\mathcal{H}_{B1}$ –sets are nothing but $\alpha-\mathcal{H}$ –opensets in quasi topological spaces.

Theorem 3.3. *Let (X,κ) be a quasi-topological space and $\mathcal{H}$ be a hereditary class of subsetsof X. Then $\mathcal{H}_{B1}(X)=\alpha\mathcal{H}o(X)=\mathcal{H}_{A1}(X)$.*

Proof: Suppose $A\in\mathcal{H}_{B1}(X)$. Then $A=U\cap V$ where U is $\alpha-\mathcal{H}$ –open and $cl_\kappa^*(\mathrm{int}_\kappa(V))=X$. Thus $V\subset X=\mathrm{int}_\kappa(cl_\kappa^*(\mathrm{int}_\kappa(V)))$, since $V\in\alpha\mathcal{H}o(X)$.

Conversely, if $U\in\alpha\mathcal{H}o(X)$, then $U=U\cap X$ where $cl_\kappa^*(int_\kappa(X))=X$ and so $U\in\mathcal{H}_{B1}(X)$. Hence $\alpha\,\mathcal{H}o(X)=\mathcal{H}_{B1}(X)$. Next suppose $A\in\alpha\mathcal{H}o(X)$. Then $A\subset int_\kappa\left(cl_\kappa^*(int_\kappa(A))\right)$ and so

$$A=int_\kappa\left(cl_\kappa^*(int_\kappa(A))\right)\cap\left(X-\left(int_\kappa\left(cl_\kappa^*(int_\kappa(A))\right)-A\right)\right)$$
$$=int_\kappa(cl_\kappa^*(int_\kappa(A)))\cap((X-int_\kappa(cl_\kappa^*(int_\kappa(A))))\cup A)$$

Also,

$$cl_\kappa^*(int_\kappa((X-int_\kappa(cl_\kappa^*(int_\kappa(A))))\cup A))$$
$$\supset cl_\kappa^*(int_\kappa(X-int\kappa(cl_\kappa^*(int_\kappa(A))))\cup int_\kappa(A))$$
$$=cl_\kappa^*\left(int_\kappa\left(X-int_\kappa\left(cl_\kappa^*(int_\kappa(A))\right)\right)\right)\cup cl_\kappa^*(int_\kappa(A))$$
$$\supset cl_\kappa^*\left(int_\kappa\left(X-cl_\kappa^*(int_\kappa(A))\right)\right)\cup cl_\kappa^*(int_\kappa(A))$$
$$\supset int_\kappa\left(X-cl_\kappa^*(int_\kappa(A))\right)\cup cl_\kappa^*(int_\kappa(A))$$
$$\supset int_\kappa\left(\left(X-cl_\kappa^*(int_\kappa(A))\right)\cup cl_\kappa^*(int_\kappa(A))\right)$$
$$=int_\kappa(X)=X.$$

Therefore, $A\in\mathcal{H}_{A1}(X)$ which implies that $\mathcal{H}_{B1}(X)=\alpha Ho(X)\subset\mathcal{H}_{A1}(X)$. Clearly, $\mathcal{H}_{A1}(X)\subset\mathcal{H}_{B1}(X)$. This completes the proof.

Theorem 3.4. *Let (X,κ) be a quasi-topological space with a hereditary class $\mathcal{H}$. If $\mathcal{H}$ is κ* –codense, then every κ –open set is a $\mathcal{H}$ –locally closed set.

Proof: Suppose that U is $\kappa-$open. By Lemma 1.3, $U\subset U^*$ and so $U^*=U^{**}$. Since $U=U\cap U^*$, implies that U is $\mathcal{H}$ –locally closed set.

The following Lemma 3.5 is useful to prove the following Theorem 3.6.

Lemma 3.5. Let (X,κ) *be a quasi-topological space with a κ –codense hereditary class $\mathcal{H}$ and A be a subset of X. If A is an $\mathcal{H}_C-$ set, then $int_{\alpha-\mathcal{H}}(A)=int_\kappa(A)$ where $int_{\alpha-\mathcal{H}}(A)$ is theinterior of A with respect to the family of all $\alpha-\mathcal{H}$ –open set $\alpha-\mathcal{H}(X)$.*

Proof: Clearly $int_{\alpha-\mathcal{H}}(A)\supset int_\kappa(A)$. Since A is an $\mathcal{H}_C$ –set, $A=U\cap V$ where U is κ –open and $int_\kappa(cl_\kappa^*(int_\kappa(V)))=int_\kappa(V)$. Now $A\subset V$ implies that $int_\kappa\left(cl_\kappa^*(int_\kappa(A))\right)\subset int_\kappa\left(cl_\kappa^*(int_\kappa(V))\right)$ $=int_\kappa(V)$.Therefore $int_{\alpha-\mathcal{H}}(A)=A\cap int_\kappa\left(cl_\kappa^*(int_\kappa(V))\right)\subset A\cap$

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$int_\kappa(V) \subset U \cap int_\kappa(V) = int_\kappa(U \cap V) = int_\kappa(A)$.
Therefore $int_{\alpha-\mathcal{H}}(A) \subset int_\kappa(A)$. Hence $int_{\alpha-\mathcal{H}}(A) = int_\kappa(A)$.

Theorem 3.6. *Let (X,κ) be a quasi-topological space with a $\kappa-$codense hereditary class $\mathcal{H}$* and A be a subset of X. Then the following are equivalent.

(a) *A is $\kappa-$open set.*

(b) *A is an $\alpha-\mathcal{H}-$open set and a $\mathcal{H}-$locally closed set.*

(c) *A is a $pre-\mathcal{H}-$open set and a $\mathcal{H}-$locally closed set.*

(d) *A is a $pre-\mathcal{H}-$open set and an $\mathcal{H}_\mathbf{A}-$set.*

(e) *A is a $pre-\mathcal{H}-$open set as well as $\mathcal{H}_\mathbf{C}-$ set and an $\mathcal{H}_\mathbf{A}-$set.*

(f) *A is a $pre-\mathcal{H}-$open set as well as $\mathcal{H}_\mathbf{C}-$set and a $semi-\mathcal{H}-open$ set.*

(g) *A is a $pre-\mathcal{H}-$open set as well as $\mathcal{H}_\mathbf{C}-$set and $int_\kappa(cl_\kappa^*(A)) \subset cl_\kappa^*(int_\kappa(A))$.*

Proof: $(a) \Rightarrow (b)$. If A is $\kappa-$open, then A is $\alpha-\mathcal{H}-$open. Since $\mathcal{H}$ is $\kappa-$codense, by Lemma 1.2, $A \subset A^*$ and so $A = A \cap A^*$. Therefore by Theorem 2.3, A is an $\mathcal{H}-$locally closed set.

$(b) \Rightarrow (c)$. Follows from the fact every $\alpha-\mathcal{H}-$open set is $pre-\mathcal{H}-$open set.

$(c) \Rightarrow (d)$. If A is an $\mathcal{H}-$locally closed set then, by Theorem 3.3, $A = G \cap A^*$ for some $\kappa-$open set G. Since $A \subset A^*$, $A^* = cl_\kappa^*(A)$. Also A is a $pre-\mathcal{H}-$open set implies that $A \subset int_\kappa\left(cl_\kappa^*(A)\right) = int_\kappa(A^*)$ and so $A^* \subset (int_\kappa(A^*))^* \subset (A^*)^* \subset A^*$. Therefore $A = (int_\kappa(A^*))^*$. Hence $A = G \cap A^*$ for some $\kappa-$open set G and $A^* = (int_\kappa(A^*))^*$ and so A is an $\mathcal{H}_\mathbf{A}-$ set.

$(d) \Rightarrow (e)$. If A is an $\mathcal{H}_\mathbf{A}-$set, then $A = G \cap V$ where G is $\kappa-$open and $V = \left(int_\kappa(V)\right)^*$. Now $int_\kappa\left(cl_\kappa^*\left(int_\kappa(V)\right)\right) = int_\kappa\left(int_\kappa(V) \cup \left(int_\kappa(V)\right)^*\right)$ $= int_\kappa(int_\kappa(V) \cup V) = int_\kappa(V)$. Therefore, A is a $\mathcal{H}_\mathbf{C}-$ set.

$(e) \Rightarrow (f)$. If A is an $\mathcal{H}_\mathbf{A}-$set, then $A = U \cap V$ where U is $\kappa-$open and $V = \left(int_\kappa(V)\right)^*$. Now $A = U \cap V = U \cap \left(int_\kappa(V)\right)^* \subset \left(U \cap int_\kappa(V)\right)^*$ $= (int_\kappa(U \cap V))^* = (int_\kappa(A))^* \subset cl_\kappa^*(int_\kappa(A))$ and so A is a $semi-\mathcal{H}-open$.

$(f) \Rightarrow (g)$. If A is a $semi-\mathcal{H}-open$, then $A \subset cl_\kappa^*(int_\kappa(A))$. Now $int_\kappa(cl_\kappa^*(A)) \subset int_\kappa(cl_\kappa^*(cl_\kappa^*(int_\kappa(A)))) = int_\kappa(cl_\kappa^*(int_\kappa(A))) = cl_\kappa^*(int_\kappa(A))$. Therefore $int_\kappa(cl_\kappa^*(A)) \subset cl_\kappa^*(int_\kappa(A))$.

$(g) \Rightarrow (a)$. Since A is $pre-\mathcal{H}-open$ set, $A \subset int_\kappa(cl_\kappa^*(A)) = int_\kappa(int_\kappa(cl_\kappa^*(A))) \subset int_\kappa(cl_\kappa^*(int_\kappa(A)))$ and so A is $\alpha-\mathcal{H}-open$ set and so $int_{\alpha-\mathcal{H}}(A) = A$. By Lemma 3.5, it follows that $int_\kappa(A) = A$ and so A is $\kappa-$open.

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References

A.Csaszar, Generalized topology, generalized continuity, Acta Math. Hungar, 96(2002), 351 -357.

A. Csaszar, Generalized Open Sets , Acta Mathematica Hungaria., 75(1-2)(1997), 65 -87.

A.Csaszar, Modifications of generalized topologies via hereditary classes, Acta Math. Hun-gar., 115(2007) 29 - 36.

A.Csaszar, Remarks on quasi-topologies, Acta Math. Hungar., 119(1-2)(2007), 197 - 200.

R.Shankar, A.P. Ponraj and D. Sivaraj Weakly semi−H−open sets and almost strong H− κ−open sets, International Journal of Pure and Applied Mathematics, 117 (17), 2017, 7-13.

A.P. Ponraj and D. Sivaraj , Semi*-H-open sets, International Journal of Pure and Ap-plied Mathematics, 117(17), 2017, 1-6.

V.Renukadevi and Sheena Scaria, On hereditary classes in generalized topological spaces, J. Adv. Res. Pure. Math., 3(2)(2011), 21-30.

V.Renukadevi and K. Karuppayi, On modifications of generalized topologies via hereditaryclasses, J. Adv. Res.Pure.Math., 2(2)(2010),14 - 20.

P.Elavarasan and A.P.Ponraj, On κ_{gp}^{*} −Closed Sets, Design Engineering 8(2021), 10994-10997